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A STUDY OF BORSUK-ULAM TYPE INEQUALITIES

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ABSTRACT. After recalling Borsuk-Ulam type theorems, we shall provide some families of Borsuk-Ulam groups. As an application, we shall show variants of the (isovariant) Borsuk-Ulam theorem for such families.

1. BORSUK-ULAM TYPE THEOREMS

K. Borsuk [2] proved the following result, called the Borsuk-Ulam theorem.

Theorem 1.1. *Let S^n be the unit sphere centered at the origin of \mathbb{R}^{n+1} , $n \geq 1$.*

- (1) *For any continuous map $f : S^n \rightarrow \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = f(-x)$.*
- (2) *If there exists an antipodal map $f : S^m \rightarrow S^n$, then $m \leq n$.*

This result was generalized in following way.

Theorem 1.2 (Free version). (1) *Suppose that a finite group G acts freely on S^m , S^n . If there exists a G -map $f : S^m \rightarrow S^n$ then $m \leq n$.*

- (2) *Let X be a path-connected free C_k -space and Y a Hausdorff free C_k -space, $k \geq 2$. If $H_q(X; \mathbb{Z}_k) = 0$ for $1 \leq q \leq m$ and $H_{m+1}(Y/C_k; \mathbb{Z}_k) = 0$ for some $m \geq 1$, then there is no continuous C_k -map $f : X \rightarrow Y$ ([7]).*

Theorem 1.3 (Fixed-point-free version [5]). *Suppose that $G = C_p^k$ acts fixed-point-freely on S^m , S^n . If there exists a G -map $f : S^m \rightarrow S^n$, then $m \leq n$.*

Moreover, T. Bartsch [1] researched Borsuk-Ulam type theorems for G -maps between fixed-point-free representation spheres for any compact Lie group G . Consequently,

Theorem 1.4 ([1]). *Let G be a finite group. The weak Borsuk-Ulam theorem holds for G if and only if G is a p -group.*

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Here “weak” means that a weaker inequality for dimensions holds, see [1] for the detail.

2. ISOVARIANT VERSION

A G -map $f : X \rightarrow Y$ is called G -isovariant if $G_{f(x)} = G_x$ for any $x \in X$. Wasserman [9] studied an isovariant version of the Borsuk-Ulam theorem. Let V, W be orthogonal representations of G , and $S(V), S(W)$ their unit spheres. The Borsuk-Ulam theorem for a free action leads to the following.

Proposition 2.1. *Let C_p be a cyclic group of prime order p . If there is a C_p -isovariant map $f : S(V) \rightarrow S(W)$, then*

$$\dim V - \dim V^{C_p} \leq \dim W - \dim W^{C_p}.$$

Definition. G is called a *Borsuk-Ulam group* (BUG) if the isovariant Borsuk-Ulam theorem holds for G -representations; namely, if there is a G -isovariant map $f : S(V) \rightarrow S(W)$, then

$$\dim V - \dim V^G \leq \dim W - \dim W^G \text{ (Borsuk-Ulam inequality)}$$

holds.

A main problem is

Problem (unsolved). *Which groups are Borsuk-Ulam groups?*

There are some partial results on this problem.

Theorem 2.2 ([9]). *If G satisfies the prime condition, then G is a Borsuk-Ulam group.*

Here we say that G satisfies the prime condition if the following condition (PC):

Let $1 = G_0 \triangleleft \cdots \triangleleft G_r = G$ be a composition series. For every simple factor $F_i := G_{i+1}/G_i$,

$$p(g) := \sum_{p: \text{prime factor of } |g|} \frac{1}{p} \leq 1$$

for any $g \in F_i$.

Example 2.3 ([9]). (1) *Solvable groups.*

(2) A_5, A_6, \dots, A_{11} (but $A_n, n \geq 12$, do not satisfy (PC).)

There is another type of example.

Theorem 2.4 ([8]). *$PSL(2, q)$ is a Borsuk-Ulam group, where q is a prime power.*

Remark. $PSL(2, 59), PSL(2, 61)$, etc do not satisfy (PC).

This result is proved as the following way. Let $G = PSL(2, q)$ and $f : S(V) \rightarrow S(W)$ be G -isovariant. Then $\text{res}_C f : S(V) \rightarrow S(W)$ is C -isovariant for any $C \in Cy(G)$, where $Cy(G)$ is the set of cyclic subgroups of G . Thus

$$\dim V - \dim V^C \leq \dim W - \dim W^C, \quad \forall C \in Cy(G)$$

$$(\because C \text{ is a BUG.})$$

$$\Downarrow (Q)$$

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

Question. Which groups satisfy implication (Q) as above?

We here call such a group G a (Q) -group.

Remark. A (Q) -group is a Borsuk-Ulam group.

Proposition 2.5 ([6]). *An abelian (Q) -group is cyclic or C_p^k for some prime p and $k \geq 0$.*

3. ALGEBRAIC DESCRIPTION OF THE BORSUK-ULAM INEQUALITY

We first recall the Möbius function. Let $S(G)$ be the set of subgroups of G .

Definition (Möbius function $\mu : S(G) \times S(G) \rightarrow \mathbb{Z}$). For $H, K \in S(G)$ with $H \leq K$, μ is inductively defined by

$$\sum_{H \leq L \leq K} \mu(H, L) = \begin{cases} 0 & H < K \\ 1 & H = K. \end{cases}$$

For $H \not\leq K$, set $\mu(H, K) = 0$.

As is well-known,

Proposition 3.1 (Möbius inversion).

$$f(K) = \sum_{H \leq K} g(H) \quad (\forall K \in S(G))$$

$$\Updownarrow$$

$$g(K) = \sum_{H \leq K} \mu(H, K) f(H) \quad (\forall K \in S(G))$$

Let V, W be G -representations and χ_V, χ_W their characters. For $H \in S(G)$,

$$\dim W^H - \dim V^H = \frac{1}{|H|} \sum_{g \in H} (\chi_W(g) - \chi_V(g))$$

$$\begin{aligned}\dim W - \dim V &= \chi_W(1) - \chi_V(1) \\ &= \frac{1}{|H|} \sum_{g \in H} (\chi_W(1) - \chi_V(1))\end{aligned}$$

Set

$$e(g) = \chi_W(1) - \chi_W(g) - \chi_V(1) + \chi_V(g)$$

and

$$h(H) = \sum_{g \in H} e(g).$$

Lemma 3.2.

$$h(H) = |H|(\dim W - \dim W^H - \dim V + \dim V^H).$$

In particular, $h(1) = 0$ and the Borsuk-Ulam inequality is equivalent to $h(G) \geq 0$. For $H \in S(G)$, we see

$$h(H) = \sum_{D \in Cy(H)} \sum_{g \in D^*} e(g),$$

where D^* is the set of generators of D . Set

$$k(D) = \sum_{g \in D^*} e(g).$$

For $C \in Cy(G)$,

$$h(C) = \sum_{D \leq C} k(D).$$

The Möbius inversion (on $Cy(G)$) says

$$k(C) = \sum_{D \leq C} \mu(D, C) h(D).$$

On the other hand,

$$h(G) = \sum_{g \in G} e(g) = \sum_{C \in Cy(G)} k(C).$$

Thus we have

$$\begin{aligned}h(G) &= \sum_{C \in Cy(G)} \sum_{D \leq C} \mu(D, C) h(D) \\ &= \sum_{1 \neq D \in Cy(G)} \left(\sum_{D \leq C \in Cy(G)} \mu(D, C) \right) h(D).\end{aligned}$$

For $D \in Cy(G)$, set

$$m(D) = \sum_{D \leq C \in Cy(G)} \mu(D, C).$$

Clearly we see

Lemma 3.3. *If $m(D) \geq 0$ for $1 \neq D \in Cy(G)$, then (Q) holds.*

The following are examples of (Q)-groups: C_n , D_n , C_p^k , $PSL(2, q)$. On the other hand, the quaternion group Q_8 is not a (Q)-group. Furthermore, there is a class of finite groups with $m(D) \geq 0$ for any cyclic $D \neq 1$.

Definition. A (P)-group G is defined to be a finite group such that any nontrivial element has prime order.

Lemma 3.4. *A (P)-group is a (Q)-group.*

Proof. $m(D) = 1$ for cyclic $D \neq 1$. □

Clearly, if a (P)-group G is abelian, then G is C_p^k for some p, k , but there are many nonabelian examples.

There are three types of (P)-groups, see [3] for the detail:

(1) p -groups with exponent p .

ex. C_p^k , $P_3 = \langle x, y, z | x^p = y^p = z^p = 1, [x, y] = z, [x, z] = [y, z] = 1 \rangle$. ($|P_3| = p^3$.)

(2) $1 \rightarrow P \rightarrow G \rightarrow C_q \rightarrow 1$ (ex) with some condition, where P is of type 1 above. (p, q are distinct primes.)

ex. $Z_{p,q}$ metacyclic group of order pq , A_4 .

(3) A_5 . This is only one nonsolvable (P)-group.

4. VARIANTS OF THE BORSUK-ULAM THEOREM

Proposition 4.1. *Let G be a (Q)-group and V, W G -representations. If there is a C -isovariant map*

$$f_C : S(V) \setminus S(V)^C \rightarrow S(W) \setminus S(W)^C$$

for every $C \in Cy(G)$, then

$$\dim V - \dim V^G \leq \dim W - \dim W^G.$$

Proof. Note that $S(V) \setminus S(V)^C$ is C -isovariantly homotopy equivalent to $S(V - V^C)$. Thus there exists a C -isovariant map

$$\bar{f}_C : S(V - V^C) \rightarrow S(W - W^C).$$

By the isovariant Borsuk-Ulam theorem for C , we obtain

$$\dim V - \dim V^C \leq \dim W - \dim W^C$$

for every $C \in Cy(G)$. The property (Q) shows

$$\dim V - \dim V^G \leq \dim W - \dim W^G.$$

□

Remark. In this case, there is not necessarily a G -isovariant map $f : S(V) \rightarrow S(W)$.

Corollary 4.2. *Let G be a (P) -group and V, W G -representations. If there is a C -map*

$$f_C : S(V) \setminus S(V)^C \rightarrow S(W) \setminus S(W)^C$$

for every $C \in \text{Cy}(G)$, then

$$\dim V - \dim V^G \leq \dim W - \dim W^G.$$

Proof. Indeed, since C has a prime order, f_C is automatically C -isovariant. \square

We finally consider the nonlinear case.

Theorem 4.3. *Let G be a (P) -group of type 1, i.e., a p -group of exponent p . Let Σ_1 and Σ_2 be mod p homology spheres with smooth G -actions. If there is a C -map*

$$f_C : \Sigma_1 \setminus \Sigma_1^C \rightarrow \Sigma_2 \setminus \Sigma_2^C$$

for every $C \in \text{Cy}(G)$, then

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G.$$

Proof (Sketch). Note the following facts.

- Facts.** (1) Σ_i^H is a mod p homology sphere (possibly empty) for any $H \leq G$ by Smith theory.
 (2) $\Sigma_i \setminus \Sigma_i^C$ is a mod p homology sphere with homological dimension $\dim \Sigma_i - \dim \Sigma_i^C - 1$.
 (3) Σ_i has a linear dimension function, i.e., there is a G -representation V_i such that $\dim \Sigma_i^H = \dim S(V_i)^H$ for any $H \leq G$ ([4]).

Since $C \cong C_p$ acts freely on $\Sigma_i \setminus \Sigma_i^C$, it follows that

$$\dim \Sigma_1 - \dim \Sigma_1^C \leq \dim \Sigma_2 - \dim \Sigma_2^C.$$

by a Borsuk-Ulam type theorem.

By Fact (3), we obtain

$$\dim V_1 - \dim V_1^C \leq \dim V_2 - \dim V_2^C.$$

This implies

$$\dim V_1 - \dim V_1^G \leq \dim V_2 - \dim V_2^G.$$

Thus

$$\dim \Sigma_1 - \dim \Sigma_1^G \leq \dim \Sigma_2 - \dim \Sigma_2^G.$$

\square

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